Deformations of Borel-fixed ideals and rational curves on the Hilbert scheme

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Hilbert schemes

"The Hilbert scheme parametrizes the subschemes of a given projective space with fixed Hilbert polynomial."

Definition

Let $\mathbb{P}^n$ be the $n$-dimensional projective space and let $p(t)$ be an admissible Hilbert polynomial. The Hilbert scheme $\text{Hilb}_{p(t)}^n$ is the parameter scheme of the family of all subschemes in $\mathbb{P}^n$ with Hilbert polynomial $p(t)$:

The Hilbert scheme has the universal property.
**Borel-fixed ideals**

**Definition**

A Borel-fixed ideal is an ideal fixed by the action of the Borel subgroup of the linear group $GL(n + 1)$ (upper triangular matrices).

Borel ideals are hard studied for their strong combinatorial properties and they are very important in the study of the Hilbert scheme because any component and any intersection of components contains at least one point corresponding to a subscheme defined by such an ideal.
Outline

1. Backgrounds
2. Borel ideals generator
3. Rational deformations of Borel-fixed ideals
4. The connectedness of the Hilbert scheme
5. New components of Hilbert schemes
6. HSC project
Monomials

- $\mathbb{K}$ algebraically closed field of characterist 0; $\mathbb{K}[x] = \mathbb{K}[x_0, \ldots, x_n]$, polynomial ring in $n + 1$ variables; $\mathbb{P}^n = \text{Proj } \mathbb{K}[x]$.
- $\preceq$ graded term ordering; we will always suppose $x_n \succ \ldots \succ x_0$.
- Monomials written with multiindex notation $x^\alpha = x_n^{\alpha_n} \cdots x_0^{\alpha_0}$.
- $\min x^\alpha = \min \{i \text{ s.t. } x_i \mid x^\alpha\}$, $\max x^\alpha = \max \{i \text{ s.t. } x_i \mid x^\alpha\}$.
- Borel increasing move: $\forall 0 \leq j < n$ such that $\alpha_j > 0$

$$e^+_j(x^\alpha) = e^+_j(x_n^{\alpha_n} \cdots x_j^{\alpha_j} \cdots x_0^{\alpha_0}) = x_n^{\alpha_n} \cdots x_{j+1}^{\alpha_{j+1}} x_j^{\alpha_j-1} \cdots x_0^{\alpha_0}.$$  
- Borel decreasing move: $\forall 0 < i \leq n$ such that $\alpha_i > 0$

$$e^-_i(x^\alpha) = e^-_i(x_n^{\alpha_n} \cdots x_i^{\alpha_i} \cdots x_0^{\alpha_0}) = x_n^{\alpha_n} \cdots x_{i-1}^{\alpha_{i-1}} x_i^{\alpha_i-1} \cdots x_0^{\alpha_0}.$$
\textbf{Monomials}

- $\mathbb{K}$ algebraically closed field of characteristic 0; $\mathbb{K}[x] = \mathbb{K}[x_0, \ldots, x_n]$, polynomial ring in $n + 1$ variables; $\mathbb{P}^n = \text{Proj } \mathbb{K}[x]$.

- $\preceq$ graded term ordering; we will always suppose $x_n \succ \ldots \succ x_0$.

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- $\min x^\alpha = \min \{ i \text{ s.t. } x_i \mid x^\alpha \}$, $\max x^\alpha = \max \{ i \text{ s.t. } x_i \mid x^\alpha \}$.

- Borel encreasing move: $\forall$ $0 \leq j < n$ such that $\alpha_j > 0$

  $$e_j^+(x^\alpha) = e_j^+(x_n^{\alpha_n} \cdots x_j^{\alpha_j} \cdots x_0^{\alpha_0}) = x_n^{\alpha_n} \cdots x_{j+1}^{\alpha_j+1} x_{j-1}^{\alpha_{j-1}} \cdots x_0^{\alpha_0}.$$

- Borel decreasing move: $\forall$ $0 < i \leq n$ such that $\alpha_i > 0$

  $$e_i^-(x^\alpha) = e_i^-(x_n^{\alpha_n} \cdots x_i^{\alpha_i} \cdots x_0^{\alpha_0}) = x_n^{\alpha_n} \cdots x_{i-1}^{\alpha_{i-1}+1} x_{i+1}^{\alpha_{i+1}} \cdots x_0^{\alpha_0}.$$

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Borel ideals

Let $\leq_B$ be the Borel partial order on the monomials of $\mathbb{K}[x]_r$ given by the transitive closure of the relations

$$e_j^+(x^\alpha) \geq_B x^\alpha \geq_B e_i^-(x^\alpha).$$

**Definition**

We will denote by $\mathcal{P}(n, r)$ the set of monomials in $\mathbb{K}[x]_r$ with the Borel partial order. $B \subset \mathcal{P}(n, r)$ is said Borel set if

$$\forall \ x^\alpha \in B \implies e_j^+(x^\alpha) \in B, \ \forall \ e_j^+.$$ 

In the case of $\mathbb{K}$ algebraically closed of characteristic 0, an ideal $I$ is Borel-fixed if the set $\{l_r\}$ of monomials in $I$ of degree $r$ is a Borel set for any $r$.

**Definition**

Let $I$ be a Borel ideal, $\{l_r\}$ the Borel set defined by $I$ in $\mathcal{P}(n, r)$ and $\{N(I)_r\} = \mathcal{P}(n, r) \setminus \{l_r\}$.

- $x^\alpha \in \{l_r\}$ is a minimal monomial if $e_i^-(x^\alpha) \notin I, \ \forall \ e_i^-$;
- $x^\beta \in \{N(I)_r\}$ is a maximal monomial if $e_j^+(x^\beta) \in I, \ \forall \ e_j^+$. 

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- $x^\alpha \in \{I_r\}$ is a minimal monomial if $e_i^-(x^\alpha) \notin I, \forall e_i^-$;
- $x^\beta \in \{N(I)_r\}$ is a maximal monomial if $e_j^+(x^\beta) \in I, \forall e_j^+$. 
Example. $\mathcal{P}(2,4)$

\begin{align*}
x_2^4 & \rightarrow x_2^3 x_1 \rightarrow x_2^2 x_1^2 \rightarrow x_2 x_1^3 \rightarrow x_1^4 \\
e_2 & \downarrow e_1^- \\
x_2 x_0^3 & \rightarrow x_2 x_1 x_0 \rightarrow x_2 x_1^2 x_0 \rightarrow x_1^3 x_0 \\
x_2 x_0^2 & \rightarrow x_2 x_1 x_0^2 \rightarrow x_1^2 x_0^2 \\
x_2 x_0^3 & \rightarrow x_1 x_0^3 \\
x_0^4 &
\end{align*}
Example. \( P(2,4) \)
Example. $P(2, 4)$
Gotzmann number

Definition

Let $p(t)$ be an admissible Hilbert polynomial. The Gotzmann number is the best upper bound for the Castelnuovo-Mumford regularity of a subscheme having Hilbert polynomial $p(t)$.

The regularity of a saturated Borel-fixed ideal equals the maximal degree of a generator and it is bounded by the Gotzmann number.

From now on, $r$ will denote the Gotzmann number and we will view any ideal $I \in \mathcal{Hilb}^n_{p(t)}$ generated in degree $r$. 
The algorithm

Theorem (Cioffi, L., Marinari, Roggero 2010)

There exists an algorithm computing all Borel-fixed ideals with assigned projective space and Hilbert polynomial.

Sketch of the proof.

Let \( I \subset \mathbb{K}[x] \) be a Borel ideal such that \( S/I \) has Hilbert polynomial \( p(t) \). For the generic linear form \( x_0 \) we have the short exact sequence:

\[
0 \rightarrow S/I(t - 1) \cdot x_0 \rightarrow S/I(t) \rightarrow S/(I, x_0)(t) \rightarrow 0;
\]

\( \Delta p(t) = p(t) - p(t - 1) \) is the Hilbert polynomial of the generic hyperplane section of \( S/I \) and the ideal \( (I, x_0) \) is still Borel-fixed.

IDEA: Recursion on the degree of the Hilbert polynomial!
Example. $\mathbb{P}^2$ and $p(t) = 2t + 4$

Gotzmann number $r = 5$. 
Example. $\mathbb{P}^2$ and $p(t) = 2t + 4$

Gotzmann number $r = 5$. $\Delta p(t) = 2t + 5 - 2(t - 1) - 5 = 2$. 

\[
\begin{align*}
  x_2^5 &\rightarrow x_2^4x_1 \\
  x_2^4x_1 &\rightarrow x_2^3x_1^2 \\
  x_2^3x_1^2 &\rightarrow x_2^2x_1^3 \\
  x_2^2x_1^3 &\rightarrow x_2x_1^4 \\
  x_2x_1^4 &\rightarrow x_1^5
\end{align*}
\]
Example. $\mathbb{P}^2$ and $p(t) = 2t + 4$

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\[
\begin{align*}
&x_2^5 \rightarrow x_2^4 x_1 \rightarrow x_2^3 x_1^2 \rightarrow x_2^2 x_1^3 \rightarrow x_2 x_1^4 \rightarrow x_1^5
\end{align*}
\]
Example. $\mathbb{P}^2$ and $p(t) = 2t + 4$

Gotzmann number $r = 5$. $\Delta p(5) = 2$. $p(5) = 14$. 
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\[
\begin{align*}
&x_2^5 \rightarrow x_2^4 x_1 \rightarrow x_2^3 x_1^2 \rightarrow x_2^2 x_1^3 \rightarrow x_2 x_1^4 \rightarrow x_1^5 \\
&\quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
&x_2^4 x_0 \rightarrow x_2^3 x_1 x_0 \rightarrow x_2^2 x_1^2 x_0 \rightarrow x_2 x_1^3 x_0 \rightarrow x_1^4 x_0 \\
&\quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \\
&x_2^3 x_0^2 \rightarrow x_2^2 x_1 x_0^2 \rightarrow x_2 x_1^2 x_0^2 \rightarrow x_1^3 x_0^2 \\
&\quad \downarrow \quad \downarrow \quad \downarrow \\
&x_2^2 x_0^3 \rightarrow x_2 x_1 x_0^3 \rightarrow x_1^2 x_0^3 \\
&\quad \downarrow \quad \downarrow \\
&x_2 x_0^4 \rightarrow x_1 x_0^4 \\
&\quad \downarrow \\
&x_0^5
\end{align*}
\]
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\[x_2^5 \rightarrow x_2^4 x_1 \rightarrow x_2^3 x_1^2 \rightarrow x_2^2 x_1^3 \rightarrow x_2 x_1^4 \rightarrow x_1^5\]
\[x_2^4 x_0 \rightarrow x_2^3 x_1 x_0 \rightarrow x_2^2 x_1^2 x_0 \rightarrow x_2 x_1^3 x_0 \rightarrow x_1^4 x_0\]
\[x_2^3 x_0^2 \rightarrow x_2^2 x_1 x_0^2 \rightarrow x_2 x_1^2 x_0^2 \rightarrow x_1^3 x_0^2\]
\[x_2^2 x_0^3 \rightarrow x_2 x_1 x_0^3 \rightarrow x_1^2 x_0^3\]
\[x_2 x_0^4 \rightarrow x_1 x_0^4\]
\[x_0^5\]
Example. $\mathbb{P}^2$ and $p(t) = 2t + 4$

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\[
\begin{align*}
&x_2^5 \rightarrow x_2^4 x_1 \rightarrow x_2^3 x_1^2 \rightarrow x_2^2 x_1^3 \rightarrow x_2 x_1^4 \rightarrow x_1^5 \\
&\quad \downarrow \downarrow \downarrow \downarrow \downarrow \\
&x_2^4 x_0 \rightarrow x_2^3 x_1 x_0 \rightarrow x_2^2 x_1^2 x_0 \rightarrow x_2 x_1^3 x_0 \rightarrow x_1^4 x_0 \\
&\quad \downarrow \downarrow \downarrow \downarrow \\
&x_2^3 x_0^2 \rightarrow x_2^2 x_1 x_0^2 \rightarrow x_2 x_1^2 x_0^2 \rightarrow x_1^3 x_0^2 \\
&\quad \downarrow \downarrow \downarrow \\
&x_2^2 x_0^3 \rightarrow x_2 x_1 x_0^3 \rightarrow x_1^2 x_0^3 \\
&\quad \downarrow \downarrow \\
&x_2 x_0^4 \rightarrow x_1 x_0^4 \\
&\quad \downarrow \\
&x_0^5
\end{align*}
\]
Example. \( \mathbb{P}^2 \) and \( p(t) = 2t + 4 \)

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Example. $\mathbb{P}^2$ and $p(t) = 2t + 4$

Gotzmann number $r = 5$. $\Delta p(5) = 2$. $p(5) = 14$. 

\[
\begin{align*}
x_2^5 & \rightarrow x_2^4 x_1 \\
x_2^4 x_0 & \rightarrow x_2^3 x_1 x_0 \\
x_2^3 x_0^2 & \rightarrow x_2^2 x_1 x_0^2 \\
x_2^2 x_0^3 & \rightarrow x_2 x_1 x_0^3 \\
x_2 x_0^4 & \rightarrow x_1 x_0^4 \\
x_0^5 &
\end{align*}
\]
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Example. \( \mathbb{P}^2 \) and \( p(t) = 2t + 4 \)

Gotzmann number \( r = 5 \). \( \Delta p(5) = 2 \). \( p(5) = 14 \).
Given a Borel ideal $I$, let us define

- $\{I_r\}_i = \{x^\alpha \in I_r \mid \min x^\alpha \geq i\}$;
- $\{N(I)_r\}_i = \{x^\beta \in N(I)_r \mid \min x^\beta \geq i\}$;

**Proposition**

Let $I$ be a Borel-fixed ideal. $I$ defines a point on $\mathcal{Hilb}_n^p(t)$ if and only if

$$|\{N(I)_r\}_i| = \Delta^i p(r),$$

where $r$ is the Gotzmann number of $p(t)$.
Can we determine deformations to pass from a Borel ideal to another preserving the Hilbert polynomial?
Goal

Can we determine deformations to pass from a Borel ideal to another preserving the Hilbert polynomial?

\[(x_2^3, x_2^2 x_1^3)\]  \[\rightarrow\]  \[(x_2^4, x_2^3 x_1, x_2^2 x_1^2)\]
Case study: constant Hilbert polynomials

**The zero-dimensional case**

Let us consider the constant Hilbert polynomial $p(t) = d$. The Gotzmann number is $d$. If $I$ is a Borel ideal defining a point in $\mathcal{Hilb}^n_d$, then $x_1^d$ belongs to the ideal because $(I, x_0) = (1)$.

$$\forall x^\beta \in \{N(I)_d\}, \quad \min x^\beta = 0.$$

**Idea**

By definition, given a Borel ideal $I$, a minimal monomial $x^\alpha \in \{I_d\}$ and a maximal $x^\beta \in \{N(I)_d\}$, both $\{I_d\} \setminus \{x^\alpha\}$ and $\{I_d\} \cup \{x^\beta\}$ are Borel sets.

How can I choose a minimal monomial $x^\alpha$ and a maximal monomial $x^\beta$, such that the ideal

$$\langle \{I_d\} \setminus \{x^\alpha\} \cup \{x^\beta\} \rangle$$

is still Borel-fixed and corresponding to the Hilbert polynomial $p(t) = d$?
The zero-dimensional case

Let us consider the constant Hilbert polynomial \( p(t) = d \). The Gotzmann number is \( d \).

If \( I \) is a Borel ideal defining a point in \( \text{Hilb}_d^n \), then \( x_1^d \) belongs to the ideal because \( (I, x_0) = (1) \).

\[
\forall x^\beta \in \{N(I)_d\}, \quad \min x^\beta = 0.
\]

Idea

By definition, given a Borel ideal \( I \), a minimal monomial \( x^{\alpha} \in \{I_d\} \) and a maximal \( x^{\beta} \in \{N(I)_d\} \), both \( \{I_d\} \setminus \{x^{\alpha}\} \) and \( \{I_d\} \cup \{x^{\beta}\} \) are Borel sets.

How can I choose a minimal monomial \( x^{\alpha} \) and a maximal monomial \( x^{\beta} \), such that the ideal

\[
\left( \{I_d\} \setminus \{x^{\alpha}\} \cup \{x^{\beta}\} \right)
\]

is still Borel-fixed and corresponding to the Hilbert polynomial \( p(t) = d \)?
Examples of problems. $(x_2^3, x_2^2 x_1, x_2 x_1^2, x_1^4)_4$
Examples of problems. \((x^3_2, x^2_2 x_1, x_2 x^2_1, x^4_1)_4\)
Examples of problems. \((x_2^3, x_2^2 x_1, x_2 x_1^2, x_1^4)_4\)
Examples of problems. \((x_2^3, x_2^2 x_1, x_2 x_1^2, x_1^4)^4\)
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Examples of problems. \((x_2^3, x_2^2x_1, x_2x_1^2, x_1^4)_4\)
Rational deformation

Theorem

Let $I \in \mathcal{H}ilb^n_d$ be a Borel-fixed ideal and let $x^\alpha, x^\beta$ be a minimal and a maximal monomial. If

1. $e_i (x^\alpha) \neq x^\beta, \forall e_i$;
2. $\min x^\alpha = \min x^\beta = 0$;

then the ideal

$$\mathcal{I} = \left\langle \{l_d\} \setminus \{x^\alpha\} \cup \{y_0 x^\alpha + y_1 x^\beta\} \right\rangle$$

has Hilbert polynomial $q(t) = \binom{n+t}{n} - d$ for every $[y_0 : y_1] \in \mathbb{P}^1$.

In the hypothesis of the previous theorem, there exists a rational curve on $\mathcal{H}ilb^n_d$ connecting the points defined by $I$ and by $J = \langle \{l_d\} \setminus \{x^\alpha\} \cup \{x^\beta\} \rangle$:

$$\begin{array}{ccc}
\mathbb{K}[x][y_0, y_1]/\mathcal{I} & \longrightarrow & \mathcal{X} \\
\downarrow & \circ & \downarrow hilb \\
\mathbb{P}^1 & \longrightarrow & \mathcal{H}ilb^n_d
\end{array}$$
Let $I \in \mathcal{Hilb}_P^n$ be a Borel fixed ideal. We construct a deformation of $I$ swapping again a minimal monomial with a maximal one... FAILURE!
Let $I \in \mathcal{Hilb}_{p(t)}^n$ be a Borel fixed ideal. We construct a deformation of $I$ swapping again a minimal monomial with a maximal one... **FAILURE!**
Let $I \in \mathcal{Hilb}_n^p(t)$ be a Borel fixed ideal. We construct a deformation of $I$ swapping again a minimal monomial with a maximal one... FAILURE!

\[
x_2^4 \rightarrow x_2^3 x_1 \rightarrow x_2^2 x_1^2 \rightarrow x_2 x_1^3 \rightarrow x_1^4
\]

\[
x_2^3 x_0 \rightarrow x_2^2 x_1 x_0 \rightarrow x_2 x_1^2 x_0 \rightarrow x_1^3 x_0
\]

\[
x_2^2 x_0^2 \rightarrow x_2 x_1 x_0^2 \rightarrow x_1^2 x_0^2
\]

\[
x_2 x_0^3 \rightarrow x_1 x_0^3
\]

\[
x_0^4
\]
Let $I \in \mathcal{H}_{p(t)}^{n}$ be a Borel fixed ideal. We construct a deformation of $I$ swapping again a minimal monomial with a maximal one... **FAILURE!**
**Attempt 2**

We construct a deformation of $I$ swapping a minimal monomial $x^\alpha$ in $\{I_r\}_j$ with a maximal one $x^\beta$ in $\{N(I)_r\}_j$ such that $\min x^\alpha = \min x^\beta = j \ldots$ **FAILURE!**
We construct a deformation of $I$ swapping a minimal monomial $x^\alpha$ in $\{I_r\}_j$ with a maximal one $x^\beta$ in $\{N(I)_r\}_j$ such that $\min x^\alpha = \min x^\beta = j \ldots$ \textbf{FAILURE!}

$$(x_3^2, x_3x_2, x_3x_1^2, x_2^4)_4 \subset K[x_0, x_1, x_2, x_3]$$
Attempt 2

We construct a deformation of $I$ swapping a minimal monomial $x^\alpha$ in $\{I_r\}_j$ with a maximal one $x^\beta$ in $\{N(I)_r\}_j$ such that $\min x^\alpha = \min x^\beta = j \ldots$ FAILURE!
We construct a deformation of $I$ swapping a minimal monomial $x^\alpha$ in $\{I_r\}_j$ with a maximal one $x^\beta$ in $\{N(I)_r\}_j$ such that $\min x^\alpha = \min x^\beta = j \ldots$ FAILURE!
We have to swap more than two monomials!

**Definition**

Let $I \in \mathcal{Hilb}_{P(t)}^n$ be a Borel fixed ideal and let $x^\alpha$ be a minimal monomial in $\{I_r\}_j$ and $x^\beta$ a maximal monomial in $\{N(I)_r\}_j$ such that $\min x^\alpha = \min x^\beta = j$. We define the set of decreasing moves over $x^\alpha$ as

$$\mathcal{F}_\alpha = \left\{ F = (e_{i_s}^-)^{\lambda_s} \cdot \ldots \cdot (e_{i_1}^-)^{\lambda_1} \mid F(x^\alpha) \in I \right\}.$$ 

Moreover

$$\mathcal{F}_\alpha(x^\alpha) = \left\{ F(x^\alpha) \mid F \in \mathcal{F}_\alpha \right\}$$

and, if any $F \in \mathcal{F}_\alpha$ is admissible on $x^\beta$,

$$\mathcal{F}_\alpha(x^\beta) = \left\{ F(x^\beta) \mid F \in \mathcal{F}_\alpha \right\}.$$
**The general case**

**Rational curve on** $\mathcal{Hilb}_p^n(t)$

**Theorem**

Let $I \in \mathcal{Hilb}_p^n(t)$ be a Borel fixed ideal and let $x^\alpha$ be a minimal monomial in $\{I_r\}_j$ and $x^\beta$ a maximal monomial in $\{N(I)_r\}_j$. If

1. $e_i^-(x^\alpha) \neq x^\beta$, $\forall \ e_i^-$;
2. $\min x^\alpha = \min x^\beta = j$;
3. $\forall F \in \mathcal{F}_\alpha$, $F$ is admissible on $x^\beta$;
4. $\{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta)$ is a Borel set;

then there exists a rational curve on $\mathcal{Hilb}_p^n(t)$ passing through the points defined by $I$ and $J = \langle \{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta) \rangle$.

**Sketch of the proof.**

We consider the family of ideals

$$\mathcal{I} = \left\langle \{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \{y_0 \ F(x^\alpha) + y_1 \ F(x^\beta) \mid F \in \mathcal{F}_\alpha\} \right\rangle, \quad [y_0 : y_1] \in \mathbb{P}^1.$$

$\mathcal{I}$ turns out to be flat and $\mathcal{I}|_{[1:0]} = I$, $\mathcal{I}|_{[0:1]} = J$. 

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Rational deformations of Borel-fixed ideals

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Example

$$(x_3^2, x_3x_2, x_3x_1^2, x_2^4) \subseteq \mathbb{K}[x_0, x_1, x_2, x_3]$$
Example

[Diagram]

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Example
Example

\[(x_3^2, x_3 x_2, x_2^3) \subset \mathbb{K}[x_0, x_1, x_2, x_3]\]
The algorithm

Require: \( I \subset K[x] \), Borel-fixed ideal.
Ensure: set of Borel-fixed ideals reachable from \( I \) by a rational deformation.

procedure \textsc{RationalDeformations}(I)
\begin{align*}
r & \leftarrow \text{Gotzmann number of the Hilbert polynomial of } K[x]/I; \\
S & \leftarrow \emptyset; \\
\text{for } i = 0, \ldots, n - 1 \text{ do} \\
\text{for all } x^\alpha \text{ minimal in } \{I_r\}_j, x^\beta \text{ maximal in } \{N(I)_r\}_j & \text{ do} \\
& \text{if } e_i^-(x^\alpha) \neq x^\beta, \ \forall e_i^- \text{ and } \min x^\alpha = \min x^\beta = j \text{ then} \\
& \quad \mathcal{F}_\alpha \leftarrow \text{set of decreasing moves on } x^\alpha; \\
& \quad \text{if any } F \in \mathcal{F}_\alpha \text{ is admissible on } x^\beta \text{ and} \\
& \quad \{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta) \text{ is a Borel set then} \\
& \quad S \leftarrow S \cup \{\langle \{I_r\} \cup \mathcal{F}_\alpha(x^\beta) \setminus \mathcal{F}_\alpha(x^\alpha) \rangle \}; \\
& \text{end if} \\
& \text{end if} \\
& \text{end for} \\
& \text{end for} \\
& \text{return } S; \\
\text{end procedure}
\end{align*}
Global glance. $\text{Hilb}_{3t+4}^3$
Both Hartshorne in his PhD thesis *Connectedness of the Hilbert scheme* (1966) and Peeva-Stillman (2005) in a paper with the same title, prove the result determining a sequence of deformations leading from any Borel ideal to the lexsegment ideal.

**Aim**

To modify the algorithm computing the rational deformations by

- adding a term ordering
- introducing a criterion to decide uniquely the monomials to swap in order to “govern the direction” of the deformation.
The modified algorithm

Require: \( I \subset \mathbb{K}[x] \), Borel-fixed ideal.
Ensure: set of Borel-fixed ideals reachable from \( I \) by a rational deformation.

procedure RATIONAL DEFORMATIONS(\( I \))
\[ r \leftarrow \text{Gotzmann number of the Hilbert polynomial of } \mathbb{K}[x]/I; \]
\[ S \leftarrow \emptyset; \]
for \( i = 0, \ldots, n - 1 \) do
\[ \text{for all } x^\alpha \text{ minimal in } \{I_r\}_j, x^\beta \text{ maximal in } \{N(I)_r\}_j \text{ do} \]
\[ \text{if } e_i^-(x^\alpha) \neq x^\beta, \forall e_i^- \text{ and } \min x^\alpha = \min x^\beta = j \text{ then} \]
\[ \mathcal{F}_\alpha \leftarrow \text{set of decreasing moves on } x^\alpha; \]
\[ \text{if any } F \in \mathcal{F}_\alpha \text{ is admissible on } x^\beta \text{ and} \]
\[ \{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta) \text{ is a Borel set} \text{ then} \]
\[ S \leftarrow S \cup \{\langle \{I_r\} \cup \mathcal{F}_\alpha(x^\beta) \setminus \mathcal{F}_\alpha(x^\alpha) \rangle \}; \]
end if
end if
end for
end for
return \( S; \)
end procedure
The modified algorithm

Require: \( I \subset K[x] \), Borel-fixed ideal; \( \preceq \), term ordering.
Ensure: the ideal reachable from \( I \) by a deformation in the \( \preceq \) direction.

\textbf{procedure} \textsc{RationalDeformations}(\( I, \preceq \))

\( r \leftarrow \) Gotzmann number of the Hilbert polynomial of \( K[x]/I \);

\begin{algorithmic}
\State \For {\( i = 0, \ldots, n - 1 \)}
\State \ForAll {\( x^\alpha \) minimal in \( \{ I_r \}_j \), \( x^\beta \) maximal in \( \{ N(I)_r \}_j \)}
\If {\( e_i^- (x^\alpha) \neq x^\beta \), \( \forall e_i^- \) and \( \min x^\alpha = \min x^\beta = j \)}
\State \( \mathcal{F}_\alpha \leftarrow \) set of decreasing moves on \( x^\alpha \);
\If {any \( F \in \mathcal{F}_\alpha \) is admissible on \( x^\beta \) and \( \{ I_r \} \setminus \mathcal{F}_\alpha (x^\alpha) \cup \mathcal{F}_\alpha (x^\beta) \) is a Borel set}
\State \( S \leftarrow S \cup \{ \langle \{ I_r \} \cup \mathcal{F}_\alpha (x^\beta) \setminus \mathcal{F}_\alpha (x^\alpha) \rangle \} \);
\EndIf
\EndIf
\EndFor
\EndFor
\State return \( S \);
\EndProcedure
\end{algorithmic}
The modified algorithm

Require: \( I \subset \mathbb{K}[x] \), Borel-fixed ideal; \( \preceq \), term ordering.
Ensure: the ideal reachable from \( I \) by a deformation in the \( \preceq \) direction.

procedure **RATIONAL\textsc{DEFORMATIONS}(I, \preceq)**

\[ r \gets \text{Gotzmann number of the Hilbert polynomial of } \mathbb{K}[x]/I; \]

\[
\text{for } i = 0, \ldots, n - 1 \text{ do}
\]
\[
\begin{align*}
x^\alpha & \gets \min_{\preceq}\{I_r\}_j; \ x^\beta & \gets \max_{\preceq}\{N(I)_r\}_j; \\
\text{if } x^\alpha & < x^\beta \text{ and } \min x^\alpha = \min x^\beta = j \text{ then} \end{align*}
\]
\[
\mathcal{F}_\alpha \gets \text{set of decreasing moves on } x^\alpha; \]
\[
\text{if any } F \in \mathcal{F}_\alpha \text{ is admissible on } x^\beta \text{ and} \]
\[
\{I_r\} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta) \text{ is a Borel set then} \]
\[
S \gets S \cup \{\langle \{I_r\} \cup \mathcal{F}_\alpha(x^\beta) \setminus \mathcal{F}_\alpha(x^\alpha) \rangle \};
\]

end if

end if

end for

return \( S \);
end procedure
The modified algorithm

Require: \( I \subset \mathbb{K}[x] \), Borel-fixed ideal; \( \preceq \), term ordering.
Ensure: the ideal reachable from \( I \) by a deformation in the \( \preceq \) direction.

procedure \textsc{RationalDeformations}(\( I, \preceq \))
\begin{align*}
  r & \leftarrow \text{Gotzmann number of the Hilbert polynomial of } \mathbb{K}[x]/I; \\
  \text{for } i = 0, \ldots, n - 1 \text{ do } & \\
  x^\alpha & \leftarrow \min_{\preceq} \{ I_r \}_j; x^\beta \leftarrow \max_{\preceq} \{ N(I)_r \}_j; \\
  \text{if } x^\alpha \prec x^\beta \text{ and } \min x^\alpha = \min x^\beta = j \text{ then } & \\
  \mathcal{F}_\alpha & \leftarrow \text{set of decreasing moves on } x^\alpha; \\
  \text{if any } F \in \mathcal{F}_\alpha \text{ is admissible on } x^\beta \text{ and } & \\
  \{ I_r \} \setminus \mathcal{F}_\alpha(x^\alpha) \cup \mathcal{F}_\alpha(x^\beta) \text{ is a Borel set then } & \\
  \text{return } \langle \{ I_r \} \cup \mathcal{F}_\alpha(x^\beta) \setminus \mathcal{F}_\alpha(x^\alpha) \rangle; \\
  \text{end if } & \\
  \text{end if } & \\
  \text{end for } & \\
  \text{return } I; & \\
\end{align*}
end procedure
The deformation graph

Definition

Given a Hilbert scheme $\text{Hilb}_{p(t)}^n$ and a term ordering $\preceq$, we define the $\preceq$-deformation graph of $\text{Hilb}_{p(t)}^n$ as the graph $(V, E)$:

- the set of vertices $V$ contains all the Borel ideals belonging to $\text{Hilb}_{p(t)}^n$;
- the set of edges $E$ contains all the rational deformations obtained with the previous algorithm.
The deformation graph

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- the set of vertices $V$ contains all the Borel ideals belonging to $\mathcal{Hilb}_{P(t)}^n$;
- the set of edges $E$ contains all the rational deformations obtained with the previous algorithm.
Definition (Cioffi, L., Marinari, Roggero 2010)

A Borel-fixed ideal $I \in \mathcal{H}ilb^n_{p(t)}$ is a hilb-segment if there exists a term ordering $\preceq$ such that

$$x^\alpha \succeq x^\beta, \quad \forall x^\alpha \in I_r, \forall x^\beta \in N(I)_r,$$

where $r$ is the Gotzmann number of $p(t)$.

Theorem

Given the Hilbert scheme $\mathcal{H}ilb^n_{p(t)}$, let $\preceq$ be a term ordering such that the hilb-segment w.r.t. $\preceq$ exists. The $\preceq$-deformation graph is a rooted tree.
hilb-segments

Definition (Cioffi, L., Marinari, Roggero 2010)

A Borel-fixed ideal \( I \in \mathcal{H}ilb_{p(t)}^n \) is a hilb-segment if there exists a term ordering \( \preceq \) such that

\[
x^\alpha \succ x^\beta, \quad \forall \ x^\alpha \in I_r, \ \forall \ x^\beta \in N(I)_r,
\]

where \( r \) is the Gotzmann number of \( p(t) \).

Theorem

Given the Hilbert scheme \( \mathcal{H}ilb_{p(t)}^n \), let \( \preceq \) be a term ordering such that the hilb-segment w.r.t. \( \preceq \) exists. The \( \preceq \)-deformation graph is a rooted tree.
The Hilbert scheme is connected.

The DegLex-deformation graph of $\text{Hilb}_{p(t)}^n$ is connected. Any ideal $J \in \text{Hilb}_{p(t)}^n$ could be deformed in a Borel-fixed one by the computation of the generic initial ideal.
The Hilbert scheme is connected.

The DegLex-deformation graph of $\text{Hilb}^n_{\rho(t)}$ is connected. Any ideal $J \in \text{Hilb}^n_{\rho(t)}$ could be deformed in a Borel-fixed one by the computation of the generic initial ideal.
Let us consider 2 rational deformations of the ideal $I \in \mathcal{Hilb}_{p(t)}^n$:

1. $\phi_1 : I \leadsto J^1 = \langle \{I_r\} \setminus \mathcal{F}_{\alpha_1}(x^{\alpha_1}) \cup \mathcal{F}_{\alpha_1}(x^{\beta_1}) \rangle$;

2. $\phi_2 : I \leadsto J^2 = \langle \{I_r\} \setminus \mathcal{F}_{\alpha_2}(x^{\alpha_2}) \cup \mathcal{F}_{\alpha_2}(x^{\beta_2}) \rangle$;

Can we perform both deformations simultaneously? Under which conditions?

If $J^3 = \langle \{I_r\} \setminus \mathcal{F}_{\alpha_1}(x^{\alpha_1}) \cup \mathcal{F}_{\alpha_1}(x^{\beta_1}) \setminus \mathcal{F}_{\alpha_2}(x^{\alpha_2}) \cup \mathcal{F}_{\alpha_2}(x^{\beta_2}) \rangle$ belongs to $\mathcal{Hilb}_{p(t)}^n$, we can consider the ideal

$$I = \left\langle \{I_r\} \setminus \mathcal{F}_{\alpha_1}(x^{\alpha_1}) \cup \left\{ y_0 F(x^{\alpha_1}) + y_1 F(x^{\beta_1}) \mid F \in \mathcal{F}_{\alpha_1} \right\} \right. \setminus \mathcal{F}_{\alpha_2}(x^{\alpha_2}) \cup \left\{ z_0 G(x^{\alpha_2}) + z_1 G(x^{\beta_2}) \mid G \in \mathcal{F}_{\alpha_2} \right\} \rightangle,$$

$[y_0 : y_1], [z_0 : z_1] \in \mathbb{P}^1$, giving a family over $\mathbb{P}^1 \times \mathbb{P}^1$ containing the points defined by $I, J^1, J^2$ and $J^3$, so they lie on a common component.
Let us consider 2 rational deformations of the ideal $I \in \mathcal{Hilb}^n_{p(t)}$:

1. $\phi_1 : I \leadsto J^1 = \langle \{l_r\} \setminus \mathcal{F}_{\alpha_1}(x^{\alpha_1}) \cup \mathcal{F}_{\alpha_1}(x^{\beta_1}) \rangle$;
2. $\phi_2 : I \leadsto J^2 = \langle \{l_r\} \setminus \mathcal{F}_{\alpha_2}(x^{\alpha_2}) \cup \mathcal{F}_{\alpha_2}(x^{\beta_2}) \rangle$;

Can we perform both deformations simultaneously? Under which conditions?

If $J^3 = \langle \{l_r\} \setminus \mathcal{F}_{\alpha_1}(x^{\alpha_1}) \setminus \mathcal{F}_{\alpha_2}(x^{\alpha_2}) \setminus \mathcal{F}_{\alpha_1}(x^{\beta_1}) \setminus \mathcal{F}_{\alpha_2}(x^{\beta_2}) \rangle$ belongs to $\mathcal{Hilb}^n_{p(t)}$, we can consider the ideal

$$I = \langle \{l_r\} \setminus \mathcal{F}_{\alpha_1}(x^{\alpha_1}) \cup \{y_0 F(x^{\alpha_1}) + y_1 F(x^{\beta_1}) \mid F \in \mathcal{F}_{\alpha_1}\}$$

$$\setminus \mathcal{F}_{\alpha_2}(x^{\alpha_2}) \cup \{z_0 G(x^{\alpha_2}) + z_1 G(x^{\beta_2}) \mid G \in \mathcal{F}_{\alpha_2}\} \rangle,$$

$[y_0 : y_1], [z_0 : z_1] \in \mathbb{P}^1$, giving a family over $\mathbb{P}^1 \times \mathbb{P}^1$ containing the points defined by $I, J^1, J^2$ and $J^3$, so they lie on a common component.
Example. $\mathcal{Hilb}^{3}_{t+4}$
Example. $\text{Hilb}_3^{3t+4}$
Example. $\text{Hilb}_3^{3t+4}$
Example. $\text{Hilb}^3_{t+4}$
One of the intentions of my PhD thesis is the design of a software for working on Hilbert schemes. To this day, there are

- about 8500 lines of java source code;
- packages to handle monomials, Hilbert polynomials, posets;
- algorithms computing Borel-fixed ideals and their deformations.

At the web page

www.dm.unito.it/dottorato/dottorandi/lella/HSC/, you can find trial versions of all the algorithms.